Four Lenses Toward Principle-Based Understanding of Granular Segregation in Dense Flows: Particle, Continuum, Statistical, and Conceptual Approaches

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This review examines the development of concepts and models describing the spontaneous spatial separation—segregation—of dense flowing granular mixtures. Although particle-scale interactions are largely understood and modern discrete element methods can resolve system dynamics in detail, a central challenge remains: prediction has advanced more rapidly than understanding. Simulation can reveal what happens; it does not, by itself, explain why. We argue that progress requires integrating four complementary approaches: particle-based simulations, continuum descriptions based on continuity, statistical methods extending kinetic theory to dissipative systems, and conceptual models that isolate dominant mechanisms. These are not competing paradigms but distinct lenses on a multiscale phenomenon. Across them, common mathematical structures recur, including Newtonian dynamics at the particle level and flux–divergence formulations governing transport. A central thesis is that conceptual models remain indispensable. By distilling mechanisms such as kinetic sieving, squeeze expulsion, and force balance, they provide explanatory structure that complements both simulation and formal theory. Recent advances applying particle-level force models of segregation illustrate the value of combining these perspectives. At the same time, each framework encounters limitations. The central challenge is therefore to identify organizing principles that connect mechanisms across scales. Granular segregation serves as a testbed for a broader question: how understanding emerges in systems where the microscopic governing laws are known, yet collective behavior remains difficult to explain.

Keywords: Granular, Segregation, Conceptual Models

1 PRELIMINARIES: The Ubiquity of Granular Segregation

Granular segregation – the spontaneous separation of mixed particles during flow – is both ubiquitous and deceptively simple to demonstrate. Nearly any difference in physical property — density, shape, friction, stiffness — can lead to segregation. Yet despite the fact that particle-scale interactions are largely understood and can now be simulated in extraordinary detail, a central paradox remains: our ability to predict has advanced more rapidly than our ability to explain. Simulation can reveal what happens; it does not, by itself, explain why.

This review takes that gap as its point of departure. We argue that progress toward a principle-based understanding of granular segregation requires integrating four complementary lenses – particle, continuum, statistical, and conceptual. These are not competing paradigms but distinct ways of organizing explanation across scales. Among them, conceptual models play a singular role: by isolating dominant mechanisms, such as kinetic sieving and squeeze expulsion, they provide the explanatory structure needed to connect detailed dynamics to general understanding. The central challenge, therefore, is not simply to compute behavior, but to identify the organizing principles that make it intelligible.

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Yet understanding is not the only imperative. The practical stakes are equally compelling: in manufacturing, segregation is often unwanted; in mining and agriculture, often desired; in geotechnical engineering, potentially devastating as large boulders accumulating at the front of an avalanching debris flow can dramatically increase the damage along its path. Documented efforts to control segregation go back more than a century [1,2] long preceding any formal theory [3].

This review addresses particulate segregation in dense granular flows, which occur across an enormous range of settings: filling and emptying of silos and bins, chutes and pipes, rotated drums and impellers, rock avalanches, geological faults, glacial till, and stream bed load transport. Dense flows range from slowly creeping deformation to rapid debris flows. The dense flow regime is conveniently characterized by the granular inertial number, $I = \dot{\gamma} d/\sqrt{P/\rho}$ [4], which captures the ratio of the microscopic inertial timescale to the macroscopic flow timescale, where $\dot{\gamma}$ is the shear rate, d the average particle diameter, P the pressure, and ρ the particle density. Dense flow is typically defined to occur in the range $10^{-3} \lesssim I \lesssim 1$.

As important as it is to delineate what this review covers, it is equally important to specify what it will not cover. Since we study the dense flow regime, segregation in both rapid granular flows, $I \gtrsim 1$, and creeping flows, $I \lesssim 10^{-3}$, are not covered. Vibrated systems, however significant, fall outside our scope. So do cohesive powders and wet granular media. Also omitted are granular media

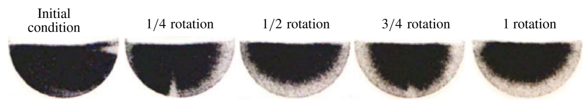

Fig. 1 Radial segregation of smaller 1 mm black particles from larger 3 mm clear particles in a half-filled circular tumbler rotating clockwise (experiments). Figure from [5] and used with permission.

fluidized by the forced flow of an interstitial fluid such as air or water.

The emphasis here is on flows driven by gravity where the necessary potential energy derives either from initial conditions (e.g., in heaps, chutes, and hoppers) or from continuous mechanical work as in a rotated circular tumbler, which serves as an ideal example of bidisperse particle size segregation. As the tumbler rotates, particles flow in a thin surface layer that is atop the bulk of particles, which rotate as a solid body. As mixed particles flow down the surface layer, small particles percolate downward and are deposited preferentially near the center of the particle bed, while large particles flow further in the surface layer to accumulate near the periphery. The result is a core of small particles surrounded by large particles at the periphery. The process is fast, occurring within only one or two tumbler rotations, as shown in Fig. 1.

Many reviews have addressed granular segregation since Williams' review in 1976 [6], which contained just eleven references. The research pace accelerated dramatically over the following two decades; by 2000, the number of references in a review on granular segregation by Ottino and Khakhar [7] had grown to over one hundred and encompassed a much broader range of materials, geometries, and approaches. Since 2000, reviews of segregation in the dense flow regime have addressed statistical and thermodynamic approaches [8], continuum approaches [9–11], bounded heap flows [12], DEM simulation [13] and relevance to pharmaceutical mixing [14]. Our review differs from all of these.

This review is being written at a moment when computational power and the discrete element method — DEM — can simulate nearly any system of interest. The particle-level physics is largely known. One could argue that the path forward is clear: simulate, observe, repeat. We argue otherwise. Simulation resolves dynamics; conceptual models resolve explanation. DEM can tell us *what* happens. It cannot, by itself, tell us *why*. And *why* is what drives understanding, generalization, and design.

This review therefore adopts a deliberate perspective: to examine how four different approaches—computational, continuum, statistical, and conceptual—contribute not only to description, but, more importantly, to understanding. These four approaches are not just for understanding granular segregation, but are broadly applicable across many mechanics subdisciplines. This review has also a deliberate historical bent — not out of nostalgia, but because the history of the field makes the case that cannot be made by simulation alone: the case for **conceptual models**. A conceptual model distills a phenomenon to its physical essence. It sacrifices completeness for insight. It is frequently *ad hoc*, grounded in a picture more than a derivation. And yet this class of models has a long and distinguished tradition in science and engineering. When successful, conceptual models reveal what is emergent and not obvious — even when the fundamental interactions are known. To quote Philip Anderson, "more is different" [15]. Conceptual models are, in this field as in others, how understanding actually advances. The central argument of this review is that conceptual models remain indispensable; that the existence and success of DEM and other computationally heavy approaches is not a reason to abandon them, but a reason to develop them more deliberately. Simulation and conceptual understanding are not substitutes; they are complements, and the field needs both.

This review is not intended to be exhaustive. Rather, it reflects a particular way of organizing and interpreting the field—one that emphasizes conceptual structure alongside methodological development. Inevitably, such a perspective brings certain contributions into focus while leaving others in the background. This selectivity is not a limitation but a consequence of our central aim: to identify organizing principles rather than to catalog results.

1.1 Granular Segregation in Context: The Long Trajectory of Foundational Ideas. What are the lines of attack to develop a principle-based understanding of granular segregation? Addressing this question in its broadest sense leads naturally to ideas that have long been central to physics — mechanics, conservation principles, and statistical descriptions designed to handle systems of increasing complexity and uncertainty. Revisiting the intellectual origins of these frameworks is not merely historical; it reveals enduring conceptual tensions that remain unresolved in the study of granular matter and, in particular, in understanding segregation. What follows is a selective tracing of the ideas whose tensions are still alive in the pages that follow.

Granular materials are, by their very nature, discrete systems. Yet whether matter itself is fundamentally discrete is a much older question. A particulate view of nature can be traced to Greek philosophers in the 5th century BCE. Leucippus and Democritus proposed that all matter is composed of indivisible units — *atomos*. In contrast, Aristotle's continuum view of matter, formulated in the 3rd century BCE, dominated scientific thought for nearly two millennia. This early dichotomy between discrete and continuous descriptions foreshadows a tension that remains central to granular materials: whether their behavior is best understood through continuum fields or through the dynamics of individual particles. In granular segregation, this is not a philosophical question but a practical one: the answer determines which models are even worth building. Or, in the cases of extreme DEM able to compute the trajectories of billions of particles (e.g., [16]) or Machine Learning (ML) powered surrogates [17], if models are even worth building.

This duality is already evident in early scientific works. Daniel Bernoulli's *Hydrodynamica* (1738) is largely framed as a continuum theory based on energy conservation in fluids. Yet, in a remarkable departure, Bernoulli introduces in its tenth chapter a primitive kinetic theory, proposing that gases consist of moving particles. This coexistence of continuum reasoning with an underlying particulate picture anticipated later developments in statistical mechanics, while highlighting a conceptual layering that persists in modern descriptions of granular flows. The same layering reappears in modern granular models: continuum transport equations carry segregation fluxes whose constitutive content must ultimately be supplied by particle-level reasoning.

A decisive reframing emerged in the 19th century with James Clerk Maxwell. Recognizing that the deterministic application of Newtonian mechanics to systems with large numbers of interacting particles is effectively intractable — owing to the extreme sensitivity and complexity of multi-particle collisions — Maxwell introduced a statistical description of molecular velocities [18]. This shift from tracking individual trajectories to characterizing ensembles marked a fundamental change in the nature of physical explanation.

The subsequent work of Boltzmann established statistical mechanics as a cornerstone of physics. However, these frameworks were developed for systems near thermal equilibrium, whereas granular materials are intrinsically out of equilibrium: energy is dissipated through inelastic collisions and friction and must be continually supplied to sustain motion. As a result, the direct application of classical statistical mechanics to granular systems is far from a simple translation. This out-of-equilibrium character is not a footnote to kinetic theory — it is the reason a direct translation fails, and why closure relations for dense granular flows remain an open problem.

The integration of these physical ideas with engineering practice introduced additional perspectives. Coulomb's work in the 18th century laid the foundations of soil mechanics by focusing on stress-induced failure and frictional behavior [19]— concepts essential for understanding stability in granular assemblies. Later, Reynolds identified dilatancy, revealing that granular materials must expand to flow, thereby linking kinematics and packing structure [20]. These contributions emphasized macroscopic behavior and practical consequences, complementing the more reductionist approaches of physics, and introduced mechanisms — friction, rearrangement, and volume change — that are central to segregation processes. Yet neither framework connected these macroscopic mechanisms to the particle-scale interactions that drive them — a gap that continues to motivate the search for constitutive relations from first principles.

Granular dynamics as a distinct discipline bridging physics and engineering began to take clearer shape in the 20th century, particularly through the work of Bagnold in the 1940s and 1950s. Bagnold systematically connected grain-scale interactions to bulk flow behavior, identifying regimes in which stresses scale with the square of shear rate and highlighting the role of particle inertia. His work marked a transition toward formulating predictive relationships grounded in particle interactions, while also underscoring the difficulty of achieving general, first-principles descriptions. The inertial scaling Bagnold identified and the rheological regimes it implies remain the foundation on which the successful granular $\mu(I)$-rheology framework [21,22] and its successors [23] are built, even as its limits define the boundaries of what those frameworks cannot explain.

Today, the study of granular segregation sits at the intersection of these intellectual threads. It involves discrete particles yet often exhibits continuum-like patterns; it is governed by mechanical interactions yet resists closure within traditional continuum frameworks; and it demands statistical descriptions while operating far from equilibrium. Moreover, segregation is inherently a multi-scale phenomenon, arising from local particle-scale mechanisms — such as percolation and kinetic sieving — interacting with global flow structures and boundary conditions. The increasing role of computation, particularly particle-based simulations such as DEM, adds a third pillar alongside theory and experiment, enabling detailed exploration while raising new questions about generalization and principle.

Against this backdrop, the central challenge is not merely to catalog phenomena and mechanisms, but to identify organizing principles that reconcile these competing descriptions. The question, then, is how to construct frameworks for granular segregation that capture its essential physics while remaining tractable across scales and levels of description.

1.2 Structure of this Review: Four Complementary Approaches. The review is organized around four complementary approaches: particle-level representations, continuum descriptions, statistical methods, and conceptual models. These are not competing paradigms but distinct lenses, each addressing the question of what constitutes an appropriate description of the system. Similar amalgams of approaches (computational or experimental, continuum, statistical, and conceptual) are not unique to descriptions of granular segregation, but apply in many other areas of research, exemplifying a common path along which science often progresses.

At the most fundamental level, three mathematical architectures recur. The first is Newton's second law applied to individual particles—the foundation of discrete simulations and force-balance models. The second is the continuity-equation framework, which underlies continuum transport equations and reaches its most general form in the Boltzmann equation. The third is cellular automata, which encode physical processes as local rules in discrete spatial and temporal spaces and can, in appropriate limits, recover continuum behavior.

Each approach offers strengths and confronts limitations. Particle methods provide detailed dynamics but require interpretation to extract general principles. Continuum approaches offer structure and, sometimes, analytical tractability but rely on constitutive relations that are often empirical and incomplete. Statistical methods provide conceptual unification but rest on assumptions that are challenged in dense, dissipative systems. Conceptual models distill mechanisms, often sacrificing completeness for insight. The first three approaches form what we call lineage-based developments—frameworks that extend established theories. Alongside them, we identify *de novo* contributions: ideas that emerged with minimal precedent and opened new directions.

1.3 Limitations of our Organizing Structure: Ideas may not Fit in Tight Boxes. Any classification imposes structure on a field that is inherently interconnected. Many contributions exhibit hybrid characteristics, combining elements of established frameworks with genuinely novel ideas. Others represent incremental advances whose cumulative impact is substantial, even if individual steps are modest. Moreover, retrospective organization can sharpen distinctions that were less clear in real time, obscuring the gradual evolution of ideas. The intellectual reality of scientific progress is inevitably more complex than any organizing framework can fully capture, and our classification cannot account for all works that contributed to the field's advancement. Recognition of these limitations reminds us that scientific advancement often occurs through pathways that resist simple categorization. Let us start by recognizing what we call *de novo* contributions.

1.4 *De Novo* Contributions: The Emergence of New Ideas . The early development of granular science is marked by a sequence of largely independent conceptual breakthroughs, each emerging under conditions of sparse prior art.

Osborne Reynolds in 1885 provided one of the first quantitative foundations through his identification of shear-induced dilatancy, where the granular volume expands to allow the particles to flow [20], establishing a key mechanical principle of granular media. His later attempt to generalize granular ideas to a universal ether in *The Sub-Mechanics of the Universe*, 1903 [24] illustrates both the reach and the speculative limits of early thinking in the absence of a mature framework.

Four decades later beginning in the 1940s, **Ralph Alger Bagnold** advanced the field with minimal theoretical inheritance. With little quantitative literature available, he relied on field observations, personal desert expeditions, and controlled experiments [25], constructing from first principles a coherent description of sand transport. Both Bagnold and his contemporaries appear to have been largely unaware of the broader impact this work would eventually have; it was presented as a careful study of sand in motion rather than as the foundation of a general framework for granular flow. His work exemplifies how, in the absence of established theory, physical intuition and experiment can jointly serve as the basis for foundational advances.

A similar pattern appears in the identification of an axial segregation instability in which alternating bands of large and small particles develop along the length of a horizontally rotated tumbler, see Fig. 2 from [26], many tumbler rotations after the initial radial segregation shown in Fig. 1. **Yositisi Oyama** (1939) first documented the phenomenon in rotating cylinders [27], yet its significance remained largely unrecognized until much later [28]. The delay reflects both limited dissemination and a lack of conceptual framing: neither the original work nor early citations emphasized band formation as a phenomenon in its own right. Instead, attention remained focused on practical concerns such as ring formation in industrial kilns, highlighting how empirical observation can precede theoretical recognition by decades.

The introduction of computational approaches by **Peter Cundall and Otto Strack** (1979) represents another independent advance [29]. Their discrete element method, primarily developed to study static stress distributions, was likewise presented with modest expectations and minimal connection to a broader theoretical program. Both the authors and discussants of Cundall and Strack's *Géotechnique* paper appear to have been largely unaware of the

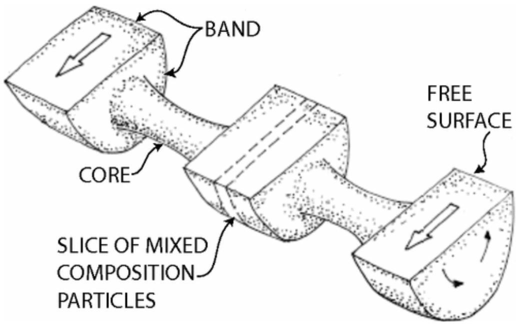

Fig. 2 Schematic illustrating location and motion (arrows) of small-particle-enriched region during axial segregation of a size-bidisperse mixture. The small particle-enriched region forms a continuous volume composed of surface bands connected by a subsurface core. Isolated bands of large particles (not shown) occupy the topologically toroidal regions outside and between the small-particle-enriched region. Figure from [26] and used with permission.

method's eventual impact on simulating flowing granular materials, as reflected in contemporaneous remarks such as "it suggests that the technique has application in investigating the mechanics of granular assemblies" and "Can the method realistically describe the behavior of real collections of particles?"

In parallel, continuum descriptions of granular flows emerged largely independently. **Peter K. Haff** (1983) formulated a fluid-like theory for granular flow, including transport equations and constitutive relations that accounted for particle-scale dissipation [30]. That closely related work by **Stuart B. Savage and David J. Jeffrey** (1981) [31] came to Haff's attention only after completion underscores the degree to which key ideas were being developed in isolation. Their analysis, particularly of rapid, dilute granular flows and collisional stresses, constitutes one of the earliest systematic continuum treatments of granular media, anticipating later kinetic-theory-based formulations, even if its broader significance was not immediately recognized.

Finally, **John Bridgwater** (1985) introduced a mathematically structured framework for segregation in granular flows [32]. Despite its understated presentation and limited citation base, the work represents a substantial conceptual leap, further reinforcing a recurring pattern: progress driven not by cumulative refinement of a shared paradigm, but by intermittent, largely disconnected advances that only later coalesce into a more unified field.

Undoubtedly, readers will identify other examples of *de novo* contributions that advanced our understanding of granular materials such as the pioneering work of **Pierre-Gilles de Gennes** (1998) on the solid/liquid transition on flowing heaps [33] and of **Sir Sam Fredrick Edwards** and colleagues [34,35] on thermodynamic analogs associated with granular packings. These examples demonstrate that departures from past approaches can begin with little precedent but open entirely new paths.

1.5 Four Lenses: Setting the Stage for Analysis. Before delving into the essential components of the paper, we first provide some background for the four main frameworks of our review:

- Particles
- Continuum
- Statistical
- Conceptual

1.5.1 The Particle Perspective: Reductionism Realized. Newton's Principia (1687) laid the foundations of modern mechanics via its three laws of motion. The 2$^{\text{nd}}$ law embodied in the differential equation $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ and its generalization to rotational motion $\tau = \frac{d\mathbf{L}}{dt}$ provide the framework for describing the evolution of the position and velocity of a body when the forces and torques acting on it are known. Exact analytic solutions were discovered early on for a variety of interaction forces including those deriving from springs (linearly elastic) and gravitational and electrostatic interactions. However, it became apparent in the mid to late 19$^{\text{th}}$ century through the works of Maxwell and Poincaré that in systems involving three or more interacting objects, general analytic solutions were not possible due to their sensitivity to initial conditions (chaos). In the case of Maxwell, this understanding was driven by the sensitivity of multiple scattering in binary collisions between hard spheres (see [36]), while in the case of Poincaré the canonical chaotic system was three objects moving under the influence of their mutual gravitational attraction and the resulting homoclinic tangle in their dynamical phase space. Early efforts to overcome the challenges of determining the time evolution of systems in the absence of analytic solutions involved a wealth of clever approximation techniques that, in the case of orbital motion, allowed the numerical calculation of orbits. In fact, a version of the Verlet integration method first employed by Jean Baptiste Delambre in 1791 to calculate orbits is now used in most numerical approaches for determining the dynamics of multi-body systems, including granular materials.

Today, in what would have been unthinkable to Maxwell, we can computationally follow large collections of particles with unprecedented precision using computer simulations [37]. This computational revolution has fundamentally altered the landscape of granular modeling, giving rise to distinct philosophical approaches that merit careful examination. Details of these advances and their application are discussed in Sec. 2.1.

1.5.2 The Continuum Perspective: Consequences of Continuity. The continuum description, physics' great workhorse, can be applied to granular systems. It is also a conceptual model in the sense that while relying on Newtonian foundations, i.e., invoking local conservation of mass, momentum, and energy, it deliberately ignores the discrete nature of matter. The advantage of the continuum view, so successful in fluid mechanics as well as many other disciplines (e.g., continuum mechanics, electricity and magnetism, and general relativity), lies in the possibilities of obtaining analytical results—a prospect that retains its allure despite the inherent graininess of the medium—and allowing relatively simple identification of dimensionless variables and their associated competing dynamics (e.g., inertial and viscous forces in the Reynolds number) from its equations of motion and constitutive relations.

Calculus and related fields as invented by Newton and Leibniz in the 17$^{\text{th}}$ century are based on the idea that quantities of interest are continuous and differentiable. Aristotle (~300 BCE) and many other natural philosophers including Leibniz would have found this hypothesis eminently reasonable when applied to material systems. Therefore, when PDEs emerged in the 18$^{\text{th}}$ century, there was little if any hesitation in putting them to work to describe the motion of matter; a notable example being Leonard Euler's work describing the dynamics of inviscid fluids (1752).

Continuity-based approaches are the fallback methodology that classical physics so often relies on and only require that certain quantities are locally conserved to set up the basic structure of the governing partial differential equations (PDEs). It is remarkably successful in describing fluid flow with the only closure needed being that of a physically realistic rheology, such as Newtonian flow where viscous stresses are linearly related to the local strain rate, independent of the flow history, velocity, and stress state. So, it is natural to apply it to granular flow, which in many ways is like fluid flow in terms of the appearance of the flow, and in requiring continuity to changes in mass, momentum, and energy. The challenge with granular flows surrounds specifying (or deriving) the constitutive relations between various quantities, which can also strongly couple the various PDEs.

Several avenues exist within the continuum framework when it comes to describing the evolution of granular mixtures which are, necessarily, composed of multiple constituent species. One earlier line of work developed in the 1970s was the branch of continuum mechanics known as mixture theory, where a system is conceived of as multiple interacting and interpenetrating continua or phases [38]. This approach assumes that at any given material point, all phases are present, and it postulates balance equations for mass and momentum for each constituent. Although this branch of research is now largely abandoned as a distinct discipline, many aspects of its approach live on in various continuum modeling approaches including those used to describe segregation in granular materials.

Euler's equations are a specific example of the differential form of the generalized continuity equation, which describes the time rate of change of a quantity q due to the actions of fluxes (flows of q per unit time per unit area) and creation (sources) or destruction (sinks) of q, that is

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{j} = \sigma, \quad (1)$$

where $\mathbf{j}$ is the flux of q, and σ, which can be positive or negative, characterizes the rate of creation or destruction, respectively, of q per unit time per unit volume. The generalized continuity equation provides a unifying thread through the theoretical frameworks encountered in this review, including, perhaps surprisingly, Boltzmann's kinetic theory. Both $\mathbf{j}$ and σ can, in general, be sums over different physical processes, i.e. $\mathbf{j} = \sum_i \mathbf{j}_i$ and $\sigma = \sum_i \sigma_i$. A flux term present in all flows is the convective flux contribution, $\mathbf{j}_c = q\mathbf{v}$. The divergence of this term as included in the continuity equation $\nabla \cdot \mathbf{j}_c = \nabla q \cdot \mathbf{v} + q\nabla \cdot \mathbf{v}$ has two components. The first term on the RHS represents the change in q due to the convective transport of gradients in q while the second term represents the changes in q due to divergence of the velocity field. In liquids and in gases at low Mach numbers, the second term is effectively zero due to the incompressibility of the flow. However, in granular mixtures with constituent particles varying in size, this assumption requires careful consideration due to the dependence of the local packing fraction on mixture composition [39], as we discuss later. A general description of granular segregation requires continuity equations for mass, momentum, and energy (granular temperature) and associated constitutive relations [30,40]. Details surrounding these choices are discussed in Sec. 2.2.

1.5.3 Statistical Mechanics: Answers from Averages. The creation of statistical mechanics offers a revealing lens through which to view contemporary approaches to granular segregation. In 1859, Maxwell formulated the first statistical law in physics — the distribution of molecular velocities in gases — fundamentally shifting the challenge from tracking individual particle trajectories to understanding average properties of large ensembles. Building on this, Boltzmann's contributions in the 1860s and 1870s, including his H-theorem of 1872, established the bridge between microscopic dynamics and macroscopic thermodynamic behavior, though they encountered considerable and surprisingly sustained resistance from the classical mechanics community.

Poincaré's work on the three-body problem in the 1880s and 1890s revealed chaos hidden within classical mechanics: deterministic systems could exhibit unpredictable behavior, providing theoretical grounding for the molecular chaos hypotheses of Maxwell and Boltzmann. Gibbs, working in parallel, consolidated the foundations of statistical mechanics in his 1902 masterwork *Elementary Principles in Statistical Mechanics* [41]. Maxwell, Poincaré and Gibbs persuaded themselves that abandoning individual trajectories in favor of a distribution function was not an approximation of convenience but a physically legitimate reframing. Maxwell's

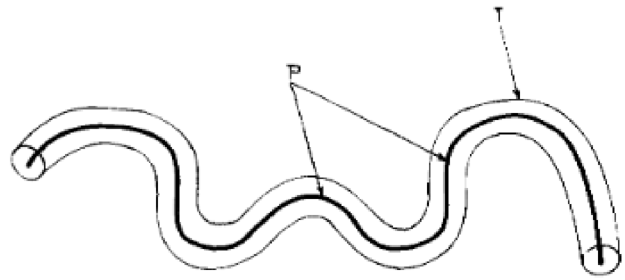

Fig. 3 de Gennes's sketch of a polymer chain moving within an effective tube formed by the surrounding polymers. Figure from [42] and used with permission (license pending).

drawings of colliding spheres [18] were argument, not illustration: arbitrarily small initial differences grow without bound, making individual trajectories epistemically irrelevant to macroscopic behavior. Poincaré gave this intuition a rigorous mathematical foundation, showing that chaos was the secret engine driving statistical mechanics — the mathematical fact that justified Boltzmann's *Stosszahlansatz* (molecular chaos assumption). Without chaos there is no ergodicity; without ergodicity there is no justification for treating a distribution function as time-independent. Gibbs completed the reframing by making the distribution function itself the primary object of mechanics: his ensemble evolves according to Liouville's equation, a flux-divergence formulation in 6N-dimensional phase space. This is the philosophical ground on which the Boltzmann equation for granular flows stands.

This progression reveals the continuity of ideas leading to domains that now appear wholly self-contained. Statistical mechanics emerged not as an inevitable development but through conceptual revolutions — much as granular physics challenges us today. In applying its ideas to granular materials, it proves useful to connect with its most developed aspect: the kinetic theory of gases. Yet this connection is far from trivial. A fundamental challenge arises from the dissipative nature of granular interactions — energy loss during collisions — which differentiates them from the energy-conserving interactions of classical statistical mechanics. Addressing this challenge is the goal of most statistical mechanics-based approaches to understanding granular segregation, as discussed in Sec. 2.3.

1.5.4 Conceptual Models: The Art of Essential Abstraction. Conceptual models have a long and distinguished tradition in science and engineering. Conceptual models are not merely pedagogical; they are often the first vehicles of discovery, revealing emergent structure before formal theory exists. When successful, they expose what is not obvious—even when the underlying interactions are known—and provide a scaffold upon which more complete theories may later be constructed.

Conceptual models often begin with images that extract the essential aspects of a problem. In some cases, these pictures remain primarily heuristic, aiding understanding without direct translation into formal mathematical frameworks. In others, they lead to simplified but powerful formulations capable of generating testable predictions. A canonical example is de Gennes's reptation model [42], where the motion of a polymer chain constrained by surrounding chains is reduced to effectively one-dimensional motion along its contour, yielding insight into relaxation dynamics, see Fig. 3. Similarly, the Ising model of magnetic spin ordering in solids [43]—despite its extreme abstraction to binary spins with nearest-neighbor interactions—captures the essence of collective behavior and phase transitions.

Some conceptual models serve as bridges between regimes—retaining elements of an established framework while introducing ad hoc constraints to capture otherwise inaccessible phenomena.

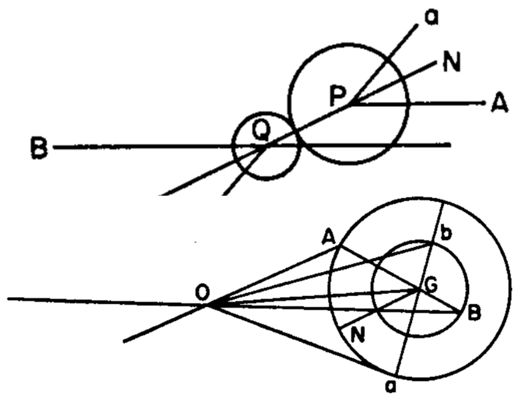

Fig. 4 Maxwell's sketches of colliding gas molecules visualized as particles. Figure from [18].

Bohr's model of the atom is a prototypical example: it combines classical orbital motion with imposed quantization rules, achieving striking predictive success despite its internal inconsistencies. Such models enable progress in situations where no fully consistent theory yet exists, functioning as intermediaries between incompatible descriptions.

Conceptual models are diverse but may be grouped into a small number of types: minimal interaction models strip a system to its simplest rules to reveal emergence from simplicity (e.g., Ising); mechanism-extracting models isolate a dominant physical process to identify rate-limiting behavior (e.g., reptation); bridging or hybrid models span incompatible theoretical frameworks (e.g., Bohr); phenomenological particle-in-medium models describe the behavior of an element embedded in an effective environment, yielding intuition in settings where full theoretical closure is not available (e.g., Jing et al. [44]). This classification is not exhaustive, but it clarifies the distinct roles conceptual models play in advancing understanding.

In contrast, Maxwell famously employed mechanical analogies as a means of persuasion only, e.g., see Fig. 4, ultimately demonstrating that such pictures were unnecessary [18]. With the subsequent development of kinetic theory and statistical mechanics, the discipline became largely picture-free: Boltzmann's and Gibbs's foundational works rely far more on formal structure than on visualization. This is not a unique case; there is a recurring arc in intellectual history: intuition gives rise to formalism, and formalism, once perfected, hides original intuition. Newton's *Principia* was geometry without equations; a century later, Lagrange's *Analytical Mechanics* prided itself on the absence of diagrams. In gaining analytical power, the field lost its pictures. This trajectory—away from pictures toward abstraction—while common, is not universal.

Granular matter provides a notable counterexample. Here, conceptual models are not merely transitional but enduring. Early descriptions of segregation, dilatancy, and kinetic sieving arise directly from physical imagery: particles rearranging, voids opening, smaller grains percolating through interstices, see Fig. 5. These approaches typically reason at the level of individual particles within an evolving local environment, using simplified representations to capture dominant tendencies without resolving the full complexity of the system.

More broadly, granular systems illustrate a domain in which conceptual modeling is structurally necessary. Unlike in classical statistical mechanics, where increasing formalism eventually supplanted pictorial reasoning, granular media continue to resist complete first-principles closure. As a result, physically grounded, picture-driven models remain central to both intuition and analysis. In this sense, they are not precursors to theory but an integral and persistent mode of reasoning.

Related examples appear across disciplines. Prandtl's mixing length theory replaces unresolved turbulent fluctuations with an effective length scale, capturing dominant transport behavior without resolving the full flow [46]. Schelling's segregation model [47], though originating in social science, demonstrates how simple local rules can generate large-scale separation, providing a conceptual parallel to physical segregation phenomena. Similarly, lamellar models of mixing reduce complex fluid deformation to the stretching and folding of material surfaces, offering a tractable representation of scalar mixing processes [48].

In this review, rather than isolating conceptual models as a separate category, we return to them in context—showing how they operate within particle-based, continuum, and statistical descriptions, and how their role evolves as one moves between these levels of representation.

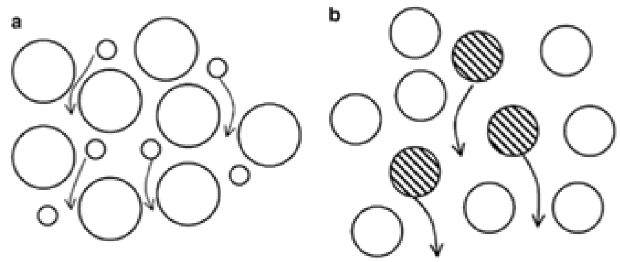

Fig. 5 Conceptual sketches of gravity-driven segregation for granular mixtures varying in (a) size and (b) density. Figure from [45] and used with permission.

2 UNDERSTANDING GRANULAR SEGREGATION: Progress from Four Perspectives

The early development of granular research shows a dual heritage. On one side, sophisticated frameworks build upon centuries of continuum and statistical mechanics, extending long intellectual lineages. On the other, bold departures begin with little precedent but open entirely new pathways. Both modes of progress—intellectual lineage and pioneering rupture—continue to shape how we approach the modeling of segregation in granular systems. We are now ready to review the conceptual approaches used to understand granular segregation.

2.1 The Particle Perspective: Reductionism Realized.

Granular systems are governed by fully deterministic laws. Every particle obeys Newton's equations of motion. Yet this determinism does not translate into predictability: the long-time evolution of many-particle systems is effectively unknowable in practice due to the chaotic amplification of collisions, as already intuited by Maxwell.

This tension between deterministic laws and practical unpredictability defines the particle perspective: rather than seeking closed-form analytical solutions, one follows the trajectories of individual particles directly through computation.

In this sense, granular materials offer perhaps the most literal realization of reductionism. The system is decomposed into its constituent particles, each governed by well-defined equations of motion, and macroscopic behavior emerges from their collective interactions. Yet this reductionist program is not merely philosophical—it is computational. The feasibility of this approach rests on the ability to simulate large numbers of interacting particles with sufficient fidelity.

Granular simulations differ fundamentally from their molecular and astrophysical counterparts in one key respect: interactions are predominantly local. Forces arise through direct contact rather than long-range fields such as electromagnetism or gravity. This locality simplifies some aspects of modeling while complicating others, particularly in dense systems where enduring contacts and force chains dominate.

Once a computational framework is adopted, three interrelated challenges come to the fore: how to model particle interactions and particle geometry, how to achieve computational efficiency at scale, and how to extract physical insight from the resulting high-dimensional data. These challenges—and the strategies developed to address them—define the evolution of discrete approaches to granular matter.

2.1.1 Hard Sphere Models. The first computational studies examining the complex dynamics of numerous particles interacting via collisional contact were performed by Berni Alder and Thomas Wainwright in the 1950's to understand emergent average dynamics in molecular systems. Rather than attempting to capture the complexity of molecular interactions, their hard sphere simulations used what is arguably the simplest model of a particle beyond Boltzmann's point particle assumption: a non-rotating sphere interacting with other identical spheres via instantaneous elastic binary collisions. Despite this simplicity, the system exhibited a liquid-solid phase transition [49] with identical results for simulations with 32 and 108 particles. Their stunning finding that such a simple interaction model with so few particles could reproduce emergent non-trivial macroscopic dynamics marked the beginning of the field of Molecular Dynamics (MD).

However, collisions between macroscopic particles, unlike the elastic hard spheres modeled by Alder and Wainwright and real molecules, are dissipative and prolonged. These differences have two important consequences for simulations, namely energy loss and the possibility of sustained non-binary contacts. To incorporate collisional energy loss, hard sphere simulations are easily adapted to describe inelastic granular materials by introducing a coefficient of restitution to characterize the ratio of the post-impact to pre-impact relative velocity of a pair of colliding particles for both the normal and tangential velocity components, which allows spin to be modeled as well. Although hard sphere simulations with instantaneous contact can faithfully model energetic rapid flows of granular materials, this approach becomes problematic when the assumption of binary collisions becomes invalid in dense granular flows (see, e.g., [50]) such as those which occur in gravity driven flows in heaps [12], chutes, tumblers, and hoppers. In these and other dense flows, energy dissipation overwhelms energy injection and inelastic collapse occurs [51,52] in which the relative velocity between particles goes to zero in a finite time, meaning they form networks of sustained contacts which stymie the hard sphere binary-collision formalism.

2.1.2 Discrete Element Method. The solution to modeling flowing granular materials with sustained multiple contacts that arise due to dissipation and elasticity came from the pioneering work of Cundall and Strack who in 1979 described the Discrete (Distinct) Element Method (DEM) (also Discrete Particle Method, DPM) to capture the "mechanical behaviour of assemblies of discs and spheres" [29]. Far from attempting to model granular flows, their original work sought to reproduce experimentally measured stress fields in quasi-statically compressed packings of polydisperse disks [53], which in turn was a simplified model for stress development in geological systems. Cundall and Strack's solution to this challenge was to model particles as elastic spheres in which forces depend on the "overlap" (to model elastic compression) between particles and calculate the resulting time evolution of particle positions and velocities by numerically integrating the equations of motion. To do so, contact forces between particles are specified along with any external forces, such those due to gravity. Their technique has proven to be extremely useful due to its wide applicability and adaptability, particularly for describing dense granular assemblies. Their original 1979 paper [29] has nearly 15000 citations at the time of writing.

Today, DEM approaches based on the work of Cundall and Strack dominate the computational landscape for granular material simulations and numerous textbooks are devoted to the topic (e.g., [54,55]). These methods numerically solve Newton's equations of motion for each particle comprising the system of interest, accounting for elastic contact forces, friction, damping and other interactions such as cohesion as well as interactions with stationary and moving boundaries. The continued progress of ever more powerful computational resources and improved algorithms make it feasible to simulate systems with billions of particles in reasonable times [16], providing the opportunity for unprecedented detail at the particle level. This brute-force computational power has made DEM a popular tool for investigating complex granular dynamics, including segregation. Numerous open-source DEM software packages are available including LAMMPS, Mercury DPM, LIGGGHTS, Yade and Chrono:GPU in addition to many commercial packages designed to model a wide range of industrial processes.

DEM simulations have been extensively validated against experiments to assess their performance and particle-level parameters, that is, to determine if they accurately reproduce the results of experiments. In most cases, validation consists of comparing flow and concentration fields at the boundaries of the granular material, e.g., at the free surface or along transparent bounding surfaces, although a wider range of 3D-imaging techniques have been developed over the past 30 years that are useful in restricted domains. These include index matching methods [56] and more exotic techniques as described in a recent focus issue [57] reviewing positron emission particle tracking, radar, magnetic resonance imaging and x-ray tomography.

A validated DEM simulation can be used as a microscope to characterize internal fluxes of segregating materials and determine their dependence on local fields, details that are challenging if not impossible to obtain from experiments alone. These dependencies can then be used to validate or devise constitutive equations as discussed in the following subsections describing continuum and statistical approaches to modeling segregation.

For mixtures of segregating granular materials composed of spheres with large to small diameter ratios, R, up to around 6 or 7, standard DEM approaches are quite capable and accurately reproduce real-world phenomena, see the example in Fig. 6. However, there are challenges where recent innovations have significantly enhanced the state of the art. One challenge associated with simulating granular materials occurs in mixtures composed of species with large size ratios. To appreciate the challenge, one must first understand how interactions between particles are calculated. The naïve approach to determine which particles interact is to compute the distance between every particle pair, an $O(N^2)$ operation, where N is the number of particles. However, because granular materials only interact by contact, it is only necessary to search for contacts in a spatially restricted local neighborhood, typically implemented by Verlet neighbor lists [58] or spatial binning (see, e.g., [54,55]) that drive the computational cost down to $O(N\log N)$. These spatial binning techniques require neighborhood dimensions on the order of the largest particle diameter, which in turn means that the maximum number of small particles in a neighborhood increases as R^3. At a size ratio of 100, there can be $O(10^6)$ particle pairs to check for contact within a single neighborhood, dramatically slowing simulations.

To overcome this large-size ratio bottleneck, elegant methods have been developed which use hierarchical grids with optimized cell sizes based on the mixture size distribution to optimally determine neighboring particles in mixtures, notably by Ogarko and Luding [60]. Implemented in Mercury DPM as well as in LAMMPS [61], these hierarchical approaches allow simulations to run with speed-ups of more than 100x for size ratios exceeding 100 compared to traditional contact detection methods [62].

In addition to size differences, segregation in dense flowing granular mixtures can be driven by differences in particle density and shape. Density is trivially controlled in the DEM approach, however modeling differences in shape can be challenging. Since nearly all DEM methods are based on interacting spheres, the sim-

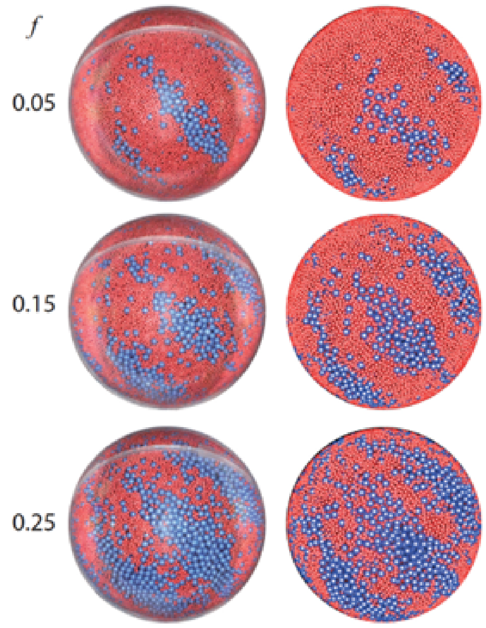

Fig. 6 Comparison of segregation patterns in a half-filled spherical tumbler of radius 7 cm rotated alternately by 57° about two orthogonal horizontal axes. Strong agreement between (left) experiment and (right) DEM simulation after 30 iterations using mixtures of 4 mm blue and 2 mm red particles for three large-particle volume fractions, *f*, (rows) viewed from below demonstrates high fidelity of DEM simulation results even in complex chaotic segregating flows. Figure from [59] and used with permission.

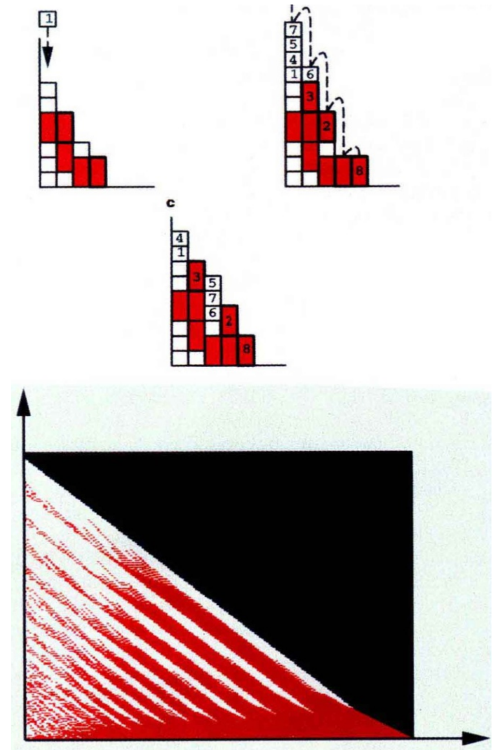

Fig. 7 Cellular automata model and resulting layered segregation for a bidisperse mixture with species differing in their repose angle. Figure from [75] and used with permission (license pending).

plest approach to modeling non-spherical particles is to construct a desired particle shape from multiple overlapping sub-spheres that move as a single rigid object. This method has various names including glued-sphere, multi-sphere and clumped sphere method. Higher shape fidelity requires more component spheres which slows computations, and modeling shapes with concavity is challenging. Other approaches to modeling non-spherical particles include the use of polyhedra [63], superquadrics/super-ellipsoids [64] as well as various hybrid approaches such as sphero-polyhedra [65] and sphero-cylinders [66]. All techniques other than the multi-sphere approach require modification of contact detection methods which are often more computationally intensive as the calculations are no longer analytical (as for spheres) but require iterative numerical approaches. However, as in many other fields, machine learning methods are being developed to speed up the process of detecting contacts and determining contact forces [67].

2.1.3 Cellular Automata. In the computational methods described to this point, physical particles are treated as distinct idealized objects characterized by a few effective parameters (e.g., diameter, restitution and friction coefficients, density) and space and time are discretized but only by the finite precision (resolution) associated with their binary numerical representations. There is another class of models, however, in which space and time are discretized such that particles typically move one unit of space in one unit of time and have the advantage of decreased computational cost and, potentially, greater conceptual clarity. This description, known as the cellular automata (CA) approach, was developed by John von Neumann and Stanislaw Ulam in the 1940s and typically involves a grid on which each distinct grid location, a cell, takes on a discrete set of states that change as dictated by a local rule that updates the state at regular intervals according to the cell's current state and the state of the cells in its local neighborhood (see [68] for a comprehensive overview). Also worth mentioning here are lattice gas automata [69,70] that combine elements of CA and Boltzmann transport formalism to describe the flow of fluids and were precursors to the more popular lattice Boltzmann methods.

Straightforward applications of CAs to granular materials are not as numerous as DEM simulations but offer distinct advantages in terms of their ability to conceptualize ideas and in their computational efficiency. CA models can potentially help to elucidate simple emergent rules that govern macroscopic segregation. Perhaps the first examples of CAs used to describe granular flow were by Bak et al. (1987) [71] and Kadanoff et al. (1989) [72], developed to better understand the concept of "self-organized criticality" and its implications for the distribution of fluctuations in many physical systems. Arguably the first CA devised to understand a realistic granular flow (silo discharge) was by Baxter and Behringer in 1990 who used a CA to capture the effects of orientable particles [73]. However, the first CA model to describe segregation was devised by Fitt and Wilmott in 1992 for a binary mixture segregating on a flowing heap [74]. Seven years later, Makse et al. (1997) described an intermittent flow layering phenomenon in quasi-2D heaps formed between parallel walls [75]. They used a CA to reproduce the observed layered segregation and intermittent flow (see Fig. 7) as well as to support their hypothesis that this was because large particles roll over smaller particles at a heap angle where small particles will not roll over large particles.

Other notable segregation studies using CA include works on tumbler blenders [76], heaps [77], and silos [78] as well as percolation of fine particles through beds of coarser particles [79], axial segregation in tumblers [80] and friction-driven segregation [81].

2.1.4 From particles to fields: Quantification via Spatial Averaging. Particle level data provided by DEM simulations fully characterizes the state and evolution of a segregating mixture. However, without further refinement it is of little value either for comparison to experiments (where particle level data is seldom available) or for revealing mechanisms that lead to understanding. Spatial averaging techniques, often referred to as coarse-graining, are essential in DEM simulations of granular segregation to convert discrete, particle-scale data, such as position, velocity, and contact forces, into continuous, macroscopic fields like concentration and velocity. By mapping individual particle data onto a structured or unstructured grid, these techniques allow quantification of the local concentration of different species, thereby enabling visualization and analysis of segregation fluxes and patterns, such as the formation of segregated cores in tumblers, and layers in unsteady heap flows. By averaging over appropriate spatial volumes and temporal intervals, the chaotic, micro-level particle data is transformed into steady, macroscopic profiles (e.g., volumetric concentration or segregation fluxes) that directly correlate with experimental findings. Spatial averaging is critical for calculating quantities like segregation velocity and diffusive flux, which are used to refine continuum models, validate statistical approaches, and reveal the underlying mechanisms, such as particle percolation, that drive segregation.

In practice, common averaging methods include bin averaging in which the domain is divided into subdomains (bins) larger than the largest particle but small enough to resolve gradients in the flow fields of interest. Refinements to this approach include accounting for the fractional volumes of a particle in each bin when the particle does not fall entirely within a single bin (e.g., [82]) and becomes particularly important for achieving accuracy when spatial variation of the velocity field occurs over length scales comparable to the particle diameter. An alternative to the spatial binning method that more naturally handles strong gradients is a particle-weighted averaging technique commonly called "coarse-graining" as developed by Babic in 1997 [83] and refined by Goldhirsch in 2010 [84]. Goldhirsch's approach mathematically resolved the challenges associated with satisfying the "continuum hypothesis," i.e., that effective averaging volumes are both sufficiently large to avoid fluctuations associated with particle discreteness and sufficiently small to resolve gradients in fields of interest (e.g., see [85]). This approach used a smooth averaging kernel to generate smooth and infinitely differentiable fields, the basis of all continuum descriptions. Weinhart et al. [86] further extended the method to properly account for boundary interactions. In general, once particle-level properties have been re-cast as continuous fields, it is possible to make the transition to the continuum perspective.

2.2 The Continuum Perspective: Consequences of Continuity. The generalized continuity equation is the standard framework for spatially extended systems — it brings the full machinery of partial differential equations to bear on transport problems by doing the "bookkeeping" associated with tracking how any locally conserved quantity can vary in time and space. Applying it to a particular material class requires specifying the velocity field as well as the physical forms for the fluxes and, in some cases, sources or sinks as found in mass transport (particle comminution and agglomeration), momentum transport (external forces due to gravitational fields and boundaries) and granular temperature transport (inelastic collisions and boundary driving).

2.2.1 Diffusive Flux. In the context of mass continuity, a flux transport mechanism common to all granular flows is the diffusive flux. Diffusion of material is often well described by Fick's law (1855), which, historically, followed in the footsteps of earlier gradient-based expressions for the diffusion of other physical quantities, namely the conduction of heat (Fourier 1807/1822) and electrical charge (Ohm 1827). Fick's law states that the diffusive flux of a quantity q is given by $\mathbf{j}_D = -D\nabla q$, where D is the diffusion coefficient. In the absence of sources or sinks of q, the continuity equation becomes:

$$\frac{\partial q}{\partial t} + \nabla \cdot (q\mathbf{v}) - \nabla \cdot (D\nabla q) = 0. \tag{2}$$

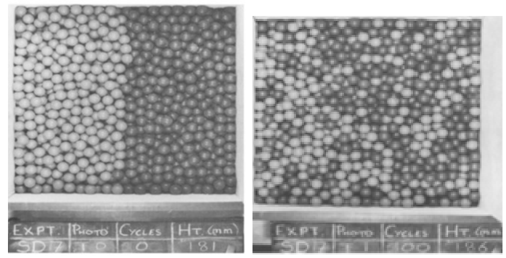

Fig. 8 Photographs from Scott and Bridgwater's pioneering experiments characterizing diffusion in dense granular flows. Figure from [87] and used with permission (license pending).

The first characterization of the diffusion coefficient in granular flow appears to have been carried out by Scott and Bridgwater in 1976 [87,88]. Their experiments used a simple shear apparatus to study the diffusive mixing of initially spatially segregated red and black spheres with diameter d under periodic shear conditions, see Fig. 8. To measure the diffusion, they stopped their experiments after different amounts of integrated strain and then carefully removed successive layers of spheres by hand, characterizing the concentration field of each layer. Assuming Fick's law, they found that the granular diffusion coefficient was well characterized by

$$D = Cd^2\dot{\gamma}, \tag{3}$$

where $\dot{\gamma}$ is the shear rate and C is a dimensionless constant of order 0.1. This expression emphasizes that relative velocity between particles, i.e. $d\dot{\gamma}$, is necessary to drive diffusion and that the characteristic length scale is the particle diameter d. Subsequent experimental studies in two dimensions [89] and in vertical channels [90,91] produced similar results.

Despite these early experimental efforts, questions remained: Is the diffusion coefficient in shear flows isotropic (in general $\mathbf{D}$ is a tensor), and does C depend on quantities such as the restitution coefficient and the volumetric packing fraction, ϕ? These questions were largely answered by Campbell in 1997 using hard sphere simulations in constant shear rate flows for ϕ up to 0.6 [50]. He found that for ϕ in the dense flow regime, the magnitude of the diffusion tensor was nearly constant, differences between diagonal components of the diffusion tensor were small (diffusion was nearly isotropic), and off-diagonal components were nearly an order of magnitude smaller than diagonal components. Further he found that the dependence on restitution coefficient was also small in the dense flow regime as expected for dense flows with persistent contacts between particles. These results imply that it is reasonable to treat diffusion as a scalar quantity in the dense flow regime.

One question that Campbell's studies did not address was the influence of pressure on the segregation rate. Given that Eq. 3 is dimensionally correct, any possible pressure dependence would have to be scaled by another stress, presumably related to particle stiffness. To address this question, Fry et al. carried out a series of DEM studies in linear shear flows, characterizing the diffusion using the mean squared displacement [92]. They found that the diffusion coefficient is independent of pressure and $C = 0.04$ even for pressure and particle stiffness combinations for which ϕ varied substantially (even beyond the random close packed limit for hard monodisperse spheres of $\phi \approx 0.64$).

2.2.2 Mixtures. To this point, we have considered the granular transport equation for a single component system. For a multidisperse incompressible mixture containing a finite number of species (discrete size distribution), the mixture theory formalism of interpenetrating continua gives rise to a set of coupled PDEs called the Advection-Diffusion-Segregation (ADS) equations [10,32,93]:

$$\frac{\partial q_i}{\partial t} + \nabla q_i \cdot \mathbf{v} + \nabla \cdot \mathbf{j}_i^{seg} - \nabla \cdot (D \nabla q_i) = 0, \qquad (4)$$

where $\mathbf{j}_i^{seg} = q_i \mathbf{v}_i^{seg}$ characterizes the segregation flux of species i and $\mathbf{v}_i^{seg}$ is the segregation velocity of species i relative to the bulk velocity $\mathbf{v}$, i.e., $\mathbf{v}_i = \mathbf{v} + \mathbf{v}_i^{seg}$. Coupling between the different PDEs for q_i can occur via dependence of the segregation velocities and diffusion coefficient on the other components q_j.

Before discussing the species dependence of the flux and diffusion terms, we first specify that the quantity of interest we will use to track the local amount of any species is the volume concentration, c_i, although other choices such as the species-specific volume fraction, ϕ_i, and the species bulk density ρ_i are also frequently used alternatives that are easily converted between. One convenience associated with using the volume concentration is that $\sum_i c_i = 1$. For bidisperse systems, the concentrations are not independent of each other as $c_1 = 1 - c_2$ and there is, consequently, only a single PDE that needs to be solved.

The expression for the diffusion coefficient of a single species is easily extended to a granular mixture. As demonstrated by Fan et al. [82] and Fry et al. [92] using DEM simulations, the original expression proposed by Bridgwater, Eq. 3, continues to accurately describe diffusion fluxes in mixtures when the particle diameter, d, is replaced by the mean local particle diameter $\bar{d} = \sum_i c_i d_i$. However, since $\bar{d}$ depends on c_i for size-disperse mixtures, the PDEs are coupled via the dependence of the diffusion coefficient on the local concentration of each species such that a mixture of N distinct species is described by $N - 1$ coupled PDEs.

2.2.3 Segregation Flux. The biggest challenge in implementing the ADS equation for granular segregation is devising an accurate and widely applicable constitutive relation for the segregation velocity and, by trivial extension, the segregation flux. As pointed out by Bridgwater et al. in 1985, certain common features for an expression describing granular segregation are widely assumed to be necessary [32] and are incorporated in nearly all expressions for the segregation flux. The first is that $\mathbf{j}_i^{seg} = 0$ when $c_i = 1$. In other words, if only a single species is present there will be no segregation flux. The second is that $\mathbf{v}_i^{seg}$ increases monotonically with decreasing c_i and is maximum when $c_i \to 0$ (though this is not true for mixtures where the species vary both in size and density [94]). It is worth noting that although $\mathbf{v}_i^{seg}$ is assumed to be maximal as $c_i \to 0$, the associated segregation flux goes to zero in the same limit. The third assumption is that the segregation velocity increases with the degree of flow agitation, where agitation is characterized in many ways via shear rate, shear rate gradient, pressure gradient, granular temperature, and kinetic stress [11]. The first three means of characterizing flow agitation (shear rate, shear rate gradient, and pressure gradient) are known once the velocity field is characterized (first two terms) and the pressure field specified (typically due to a boundary condition in fully wall bounded flow or to gravity-induced lithostatic forces in free surface flows), whereas granular temperature and kinetic stress requires knowledge of the velocity fluctuations which are not used directly in the ADS formalism.

As an example, consider an early (perhaps the first) expression for the segregation velocity in a size-bidisperse mixture as proposed by Bridgwater et al. in 1985 [32] in the context of an ADS formalism: $v_i^{seg} = v_{c_i=0}^{seg} f(c_i)$, where $v_{c_i=0}^{seg}$ is the single particle (intruder) limit segregation velocity with an unspecified shear rate dependence. To illustrate examples of solving the resulting transport equation, Bridgwater et al. discuss two possibilities for $f(c_i)$, namely $f(c_i) = (1 - c_i)^\alpha$ with $\alpha = 1$ or 2 while considering flows with constant shear rate such that $dv_{c_i=0}^{seg}/dz = 0$. The associated segregation flux is $c_i \mathbf{v}_i^{seg}$ or

$$\mathbf{j}_i^{seg} = B \mathbf{v}_{c_i=0}^{seg} c_i (1 - c_i)^\alpha, \qquad (5)$$

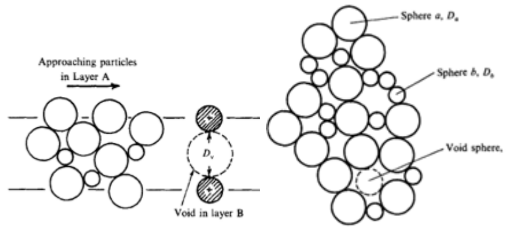

Fig. 9 (left) Top view of the kinetic sieving model of Savage and Lun. Percolation of a particle in upper layer A downward (into the plane of the page) into lower layer B depends on the void diameter, D_v, in layer B (from Fig. 3 of [95]). (right) Example random continuous network in layer B used to determine void diameter distribution (from Fig. 2 of [95]). Figures used with permission (license pending).

where B is a dimensionless constant dependent on the particle diameters and possibly other particle properties (stiffness, friction, restitution) and flow properties (pressure, volume fraction, etc.).

Many other models for $\mathbf{j}_i^{seg}$ have been proposed over the last 40 years and are summarized in a recent review by Thornton et al. for size-bidisperse mixtures [11]. Here we briefly discuss a few of these.

Kinetic Sieving. An expression for the segregation flux was derived by Savage and Lun in 1988 [95] just three years after Bridgwater et al.'s pioneering work. Their paper which developed the kinetic sieving model is an example of a **conceptual leap** in which a simple picture provides great insight and can be further developed into a mathematical model. Specifically, Savage and Lun postulated a model for the segregation velocity in a size-bidisperse dense sheared mixture (hereafter referred to as the SL model) based on the assumption that the probability for a small particle in upper layer A in Fig. 9 to fall into a void opening in lower layer B is larger than the corresponding probability for a large particle. Consequently, the unequal downward rate of void filling induces segregation of the small particles relative to large particles, with small particles segregating below the large ones. In addition to this "random kinetic sieving" mechanism, the SL model also includes a non-size-preferential term to allow the upward movement of particles, called "squeeze expulsion," which balances the downward flux of both species owing to kinetic sieving. Their key insight to turn this simple picture into a mathematical model was to use a statistical model to determine the void diameter distribution which can then be used directly to calculate the probabilities for large or small particles to fall into a void. Specifically, and following the approach of Gibbs, they used a maximum entropy approach over an ensemble of microstates of a planar random continuous network.

The mechanisms of random kinetic sieving and squeeze expulsion are additively combined in the SL model to give the net volume-averaged percolation velocity. This first-principles approach to predict particle size segregation applies in moderately sheared, gravity-driven free dense surface flows. It traces back to the information-entropy approach of Cooke and Bridgwater [96], who postulated that each particle lies in a cage defined by surrounding particles that a percolating particle must squeeze past, analogous to molecular transport in liquids and leading to the conclusion that the diameter ratio of the particle species is a key factor in determining the velocity at which species segregate from one another.

Although ground-breaking in its approach and of continuing interest as demonstrated by its citation count, the SL model has proven challenging to analyze and apply [97]. It is worth noting, however, that a model for the local segregation flux that we

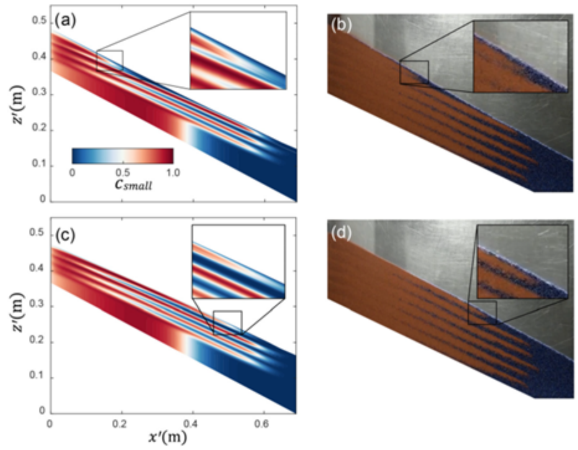

Fig. 10 Comparison of large (blue) and small (red) particle stratification in (left column) continuum model predictions and (right column) experiments [99] (right column) in a $W = 69$ cm wide and $T = 1.27$ cm thick heap under modulated feed flow conditions for an equal volume mixture of 2 mm blue and 0.5 mm red glass spheres at (a, b) 3 s into the slow-to-fast transition and at (c, d) 10 s into the fast-to-slow transition. Color bar in (a) indicates the small-particle concentration. Note that the continuum model (left column) starts with a steady feed rate (lower portion of the flow domain) followed by slightly more than three iterations of feed rate modulation, while the experiments include six iterations of feed rate modulation without an initial steady feed rate period. Figure from [98] and used with permission (license pending).

developed using DEM simulation data [82], namely

$$\mathbf{j}_i^{seg} = S\dot{\gamma} c_i (1 - c_i), \tag{6}$$

where S is a characteristic length [64], can be derived as a first order expansion in concentration of the SL model [97]. This model is applicable to many types of free surface flows, even showing good fidelity in unsteady segregating flows in tumblers [5] and in heap flows with a modulated feed rate [98], see Fig. 10, where ideas discussed by de Gennes on the relationship between local flux and the heap's neutral angle [33] are incorporated into the kinematics model.

Partial-stress Models. The partial stress method extends the mass-transport formalism by incorporating momentum transport: segregation fluxes follow from species-specific momentum equations rather than from a transport equation directly. In this approach, the momentum fluxes are determined by the unequal partial stresses acting within and between the various components of the mixture. In general, the stresses can be partitioned into gravity-induced pressure gradient contact stresses [9,100,101] associated with prolonged contacts between particles, and shear-gradient-induced kinetic stresses [102,103], which are collisional in nature. However, Staron and Philips show that computing the contact-stress partition controlling gravity-driven segregation fluxes is sensitive to the definition of the partial stress tensor, making predictions challenging [104].

2.2.4 Coupled Transport Equations. Pure ADS-framework models require as input the bulk flow kinematics from DEM simulations, experimental measurements, or a prescribed standard functional form (e.g., linear, exponential, or Bagnold velocity profile) to determine the advection, diffusion, and segregation fluxes. Similarly, partial-stress models using coupled momentum and mass transport frameworks assume that the bulk velocity field is steady and its gradients are known. However, it is possible to simultaneously solve both the momentum transport equation and the ADS equation to determine both the kinematics and the transport fluxes as they evolve in time and space. To do so and in addition to the ADS-expressions for the velocity and concentration dependent diffusion and segregation fluxes, one also needs expressions for the stress-fields that drive the velocity field.

Although various models have been proposed for the stresses, arguably the two most common methodologies are the generalized granular $\mu(I)$ rheology and the Non-local Granular Fluidity (NGF) rheology [23]. The $\mu(I)$ rheology as originally proposed by Jop, Forterre, and Pouliquen in 2006 [21] considers the shear stress in the dense flow regime to be friction-like, given by $\tau = \mu(I)P$, where I is the inertial number [22] and μ is a rate dependent friction that increases with increasing I. There are now many variants of the original $\mu(I)$ rheology that expand its applicability, but the basics remain the same. In general, the $\mu(I)$ rheology accurately models the kinematics of a wide range of dense flows in various geometries but struggles to capture the transition to quasi-static creeping flow and can suffer from numerical instabilities [105,106] at the limits of its applicability ($0.001 < I < 0.5$) associated with the assumption of granular flow incompressibility [107]. An alternative class of models with a broader range of applicability are so called non-local rheologies in which stress fluctuations, as characterized by a granular fluidity field, diffuse throughout the domain and are characterized by a particle-diameter-related length scale [108].

Several studies have coupled evolving mass and momentum fields through a rheological constitutive model. Liu et al. [109] used a finite element method approach combined with an elasto-plastic constitutive model and a free-surface flow derived segregation model [82] to simulate size-bidisperse segregation in rotating drums and discharging hoppers. Barker et al. [110] considered a modified $\mu(I)$-rheology and a pressure-gradient-driven segregation model [111] to model segregation in avalanches and non-circular tumblers. Liu et al. [112] used a non-local granular fluidity model [113,114] with shear-driven segregation. Singh et al. used both shear- and gravity-driven segregation to model cases where segregation and diffusion act both normal to the plane of shearing [115] and parallel to the plane of shearing [116].

Both the $\mu(I)$ rheology and NGF rheology approaches allow for the possibility of coupling between concentration and velocity fields through the dependence of the rheology on the local mean particle size and local packing fraction, both of which vary with concentration. However, the coupling between flow and segregation is often a secondary effect, particularly for free surface flows. In these cases, a simple expression for the segregation velocity, such as Eq. 6, along with an empirical, theoretical, computational, or experimental velocity field suffice [10].

To date, existing studies with coupled mass and momentum models have explored a small range of particle size differences ($R \lesssim 2$) where these coupling effects are relatively small, most likely because accurate segregation models for size ratios larger than around three or four are lacking. Three recent studies are of interest regarding coupling of concentration driven variation in local mean particle size and packing fraction with rheology, especially at larger size ratios where these effects are magnified.

For static beds of bidisperse particles, the volume fraction can increase by over 30% at R=10 and by over 15% at a moderate size ratio of R=3 when the particle concentration is varied from monodisperse to approximately 30% small particles for a size-bidisperse mixture [117]. This work clearly demonstrates that assumptions concerning incompressibility of the bulk flow are likely to be inaccurate for larger size ratios. Additionally, rheologies are also likely to change significantly with these large changes in ϕ, especially on the low concentration side ($c_s \lesssim 0.3$) where small particles rattle in the gaps between large particles and the high concentration side ($c_s \gtrsim 0.3$) where large particles do not interact with each other because they are in a sea of small particles.

Of recent interest is work by d'Ortona et al. showing that axial segregation bands in long tumblers [118] and longitudinal rolls in chute flows [119] appear to be driven by the concentration dependence of the local density via changes in the volume fraction. This Rayleigh-Taylor-like phenomenon should be amenable to continuum descriptions once the concentration dependent density is included in the momentum transport equation.

Also of relevance is recent work on the segregation velocity of smaller intruder particles (zero concentration limit) in uniform shear flows by Gao et al. [120]. For $3 < R < 6$ the segregation velocity is non-monotonic in the shear rate, increasing initially as geometrically trapped particles are freed by shear driven rearrangements of the large particles, but then decreasing with increasing shear rate when shear induced velocity fluctuations become comparable to the fluctuations induced by gravity alone. Clearly, we are only beginning to develop a truly comprehensive continuum description of segregation in dense granular flows.

2.3 Statistical Perspective: Answers from Averages. Continuum models of granular segregation with empirically determined constitutive relations accurately predict granular segregation in many simple flows. These relations universally include fit parameters adjusted to match experiments or simulations. In contrast, the great appeal of statistical methods is that the only parameters appearing in constitutive relations describe particles and particle interactions, e.g., diameter, density, restitution and friction coefficients. Approaches based on statistical mechanics have progressed to the point where they yield granular dynamic equations that accurately describe some simple flow regimes, but a fully-general first-principles-based framework applicable to arbitrary problems of segregation in dense flows remains an ongoing challenge.

2.3.1 Kinetic Theory of Granular Gases. The relationship between molecular gases and granular particles is not a loose analogy but a shared mathematical structure. In both cases, large assemblies of particles undergo repeated interactions that render individual trajectories irrelevant and instead admit a statistical description. This is precisely the program articulated by Maxwell, Poincaré, Boltzmann, and Gibbs with respect to molecular motion: microscopic chaos justifies a distribution-function framework. In practice, the scalar distribution function $f(\mathbf{x}, \mathbf{v}, t)$, defined in the position-velocity phase space, describes the probability density of finding a particle at position $\mathbf{x}$ with velocity $\mathbf{v}$ at time t. With f in hand, the macroscopic fields of interest are simply determined, e.g., number density, n, velocity, and granular temperature:

$$n = \int f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}, \tag{7}$$

$$\mathbf{v}(r, t) = \frac{1}{n} \int \mathbf{v} f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}, \tag{8}$$

and

$$T_g = \frac{1}{2n} \int (\mathbf{v} - \mathbf{v})^2 f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}, \tag{9}$$

respectively.

Since granular flows are always out of equilibrium, the system can evolve in time and always exhibits field gradients. The time evolution of the distribution function is governed by kinematic-fluxes and particle collision in the phase space. Specifically, Boltzmann posited a transport equation formalism:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \mathbf{a} \cdot \nabla_v f = \left(\frac{\partial f}{\partial t}\right)_{\text{collisions}} \tag{10}$$

Details of particle interactions are specified in the collision term, which Boltzmann described for the case of elastic binary collisions between point particles with uncorrelated pre-collision momenta (the so-called molecular chaos assumption, or *Stosszahlansatz*). The kinetic theory of granular flow (KTGF) is a direct extension of Boltzmann's now classic approach (see, e.g., [121]). The crucial distinction from molecular systems lies in the nature of granular collisions: interactions are dissipative rather than elastic, leading to energy losses that serve to locally correlate both particle velocity and position via enhanced clustering.

Pioneering work applying statistical mechanics approaches in the form of kinetic theory to dense granular flows begins with the work of Bagnold (1954) [122] who derived an expression for shear-induced pressure. A more formal and theoretically grounded approach is attributable to Savage and Jeffrey (1981) [31], Jenkins and Savage (1983) [123], Lun et al. (1984) [124], Jenkins and Richman (1985) [125] and Lun (1991) [126]. Much of this early work is covered in detail by Gidaspow [127].

A recent paper by Berzi summarizes these and other contributions in a discussion of the current state of the art and understanding regarding the steps required to make the transition from molecular to granular flows using the Boltzmann transport equation approach [128]. The dissipative characteristic of granular collisions is captured in terms of coefficients of normal and tangential restitution and a coefficient of friction which are combined into a single effective restitution coefficient. The finite size of particles leads to volume exclusion and enhanced collision frequency, accounted for in the collision term using the Enskog approximation, which is characterized by the empirically determined and volume fraction dependent radial distribution function. Dissipation-induced correlations between particles are captured by a phenomenological correlation length that reduces the collision rate in dense flows. The KTGF description closes onto continuum equations for mass, momentum, and energy. The implication is not merely methodological but conceptual: kinetic theory and continuum mechanics are different representations of the same underlying transport structure, expressed at different levels of resolution.

The point of breakdown of the KTGF approach is equally well defined. Classical kinetic theory assumes binary, instantaneous collisions—conditions that hold in dilute systems and create rate-dependent stresses. However, in sufficiently dense flows particles form networks of simultaneous, enduring contacts, interactions are governed as much by the enduring contact constraints as by collisional exchange, and stresses become shear rate independent. The assumptions underpinning the Boltzmann framework are therefore formally violated. This limit is conveniently characterized by a critical volume fraction ϕ_c that is now recognized as the friction-coefficient-dependent value of ϕ at the jamming transition following the pioneering work of Silbert in 2010 [129]. And yet, the framework retains explanatory power beyond the regime in which it is formally valid.

2.3.2 Kinetic Theory Models of Granular Segregation. Extensions of KTGF to granular mixtures (see discussion in Barajas, 1973 [130]) introduce interspecies momentum exchange—effectively a drag between particle species—which enables the derivation of segregation velocities from a balance with pressure and shear stresses. This framework is particularly effective for density-bidisperse systems, where predictive expressions can be obtained with relative ease and demonstrate good agreement with simulations (e.g., Duan et al. 2020 [131]). Its extension to size-bidisperse mixtures is more involved, but sustained progress—beginning with Jenkins and Mancini's extension of the revised Enskog-Thorne theory to binary granular mixtures [132]—continues to today. Notable contributions of this approach to binary mixtures include Hsiau and Hunt's derivation of purely-diffusion driven segregation in a shear flow [91], size and density-driven segregation in inclined flows for both steady-state flow [133] and evolving flow [134–136]. These examples demonstrate that, even beyond its formal limits, the kinetic-theory framework remains a generative and unifying tool.

3 FROM FRAMEWORKS TO MECHANISMS: Emergence, Structure, and Organization

The preceding sections have examined granular segregation through three established frameworks—particle, continuum, and statistical—each offering a distinct but necessarily partial account of the phenomenon. Taken individually, these approaches describe how segregation occurs; taken together, they begin to suggest why. The central challenge, however, is not the refinement of any single framework, but the identification of mechanisms that persist across them.

Understanding does not reside within a given representation. It emerges in the transitions between representations—when particle dynamics are recast as fields, when continuum fluxes are traced back to microscopic interactions, and when statistical descriptions are connected to physical mechanisms. It is in these translations that structure becomes visible.

This section therefore shifts the focus from frameworks to mechanisms. We first examine how kinematics organizes into structure, revealing persistent features of segregation across systems. We then present an integrative example in which conceptual models and simulation converge, illustrating how explanation arises when different approaches are brought into alignment.

3.1 From Kinematics to Structure: Mechanisms Made Visible. Across the frameworks reviewed in Section 2, segregation is most often described in terms of kinematics—particle trajectories, velocity fields, and fluxes. Yet kinematics alone does not constitute explanation. These structures—segregation profiles, stratification patterns, band formation, and concentration gradients—are not incidental outcomes of specific models; they are the observable signatures of processes such as kinetic sieving, squeeze expulsion, and force imbalances.

The key step, therefore, is not to refine kinematic description, but to interpret it: to recognize which features of the flow are contingent and which are invariant across representations. When similar structures arise in particle simulations, continuum models, and experiments, they signal the presence of mechanisms that transcend any single framework. In this sense, structure is the meeting point between description and explanation—it is where mechanisms become visible.

Segregation manifests across a wide range of granular flows, producing outcomes that range from weak concentration gradients to sharply organized, persistent structures. A useful point of departure is the simplest, and in many ways canonical, case: radial segregation in a two-dimensional circular container that is partially filled with size-bidisperse particles and rotated at constant speed. This configuration establishes a baseline against which more complex behaviors can be understood.

In a circular tumbler operating in the rolling regime [137], the flow consists of a thin, continuously sheared surface layer atop a solid-body rotating bulk that feeds particles into the upstream region of flowing surface layer and extracts them in the downstream region. The resulting segregation structure—a small-particle core surrounded by a large-particle annulus for tumblers half-filled or less (see Fig. 1)—is robust, reproducible, and largely insensitive to initial conditions and details of the granular mixture; dense and less dense materials, mixtures of particle shapes, etc., all lead to the same result. The kinematics are simple and effectively integrable: particle trajectories are regular, and segregation acts as a systematic drift superimposed on an otherwise predictable flow. This case exemplifies a regime in which geometry and forcing produce a steady, non-chaotic flow after a short transient and segregation outcomes are correspondingly simple and strongly constrained.

Departures from this baseline arise when the underlying kinematics become more complex. In particular, chaotic advection can be generated through two distinct routes. The first is geometric: containers that deviate from circular symmetry (e.g., elliptical or polygonal cross-sections) disrupt the integrability of particle trajectories, producing flows with coexisting regular regions and

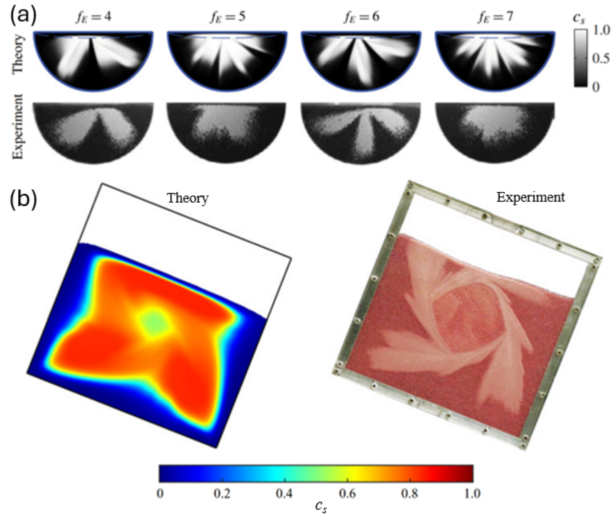

Fig. 11 Examples of radial segregation in (a) circular tumbler with modulated rotation rate showing (top) advection-diffusion-segregation theory and (bottom) experimental results (figure from [5] and used with permission) and (b) square tumbler showing (left) coupled mass-momentum continuum model theory and (right) experimental results. c_s denotes the small particle concentration. Figure from [110] and used with permission (license pending).

chaotic seas. The second is temporal: even in circular containers, time-periodic modulation of the rotation rate or axis introduces stretching and folding of material lines, again leading to chaotic advection. These two mechanisms—geometric asymmetry and time-dependent forcing—provide independent and, in many cases, complementary routes to chaos. Although reproducible in DEM simulations as well as continuum approaches, see Fig. 11 for examples, understanding these results requires conceptual ideas concerning the structure of chaotic flows and its interaction with segregation.

A large body of work has examined segregation in such time-periodic and chaotic flows, see, e.g., [7,138,139]. A central insight is that segregation breaks volume preservation. While the underlying granular flow remains effectively incompressible ($\nabla\cdot\mathbf{v} = 0$), the addition of segregation velocities alters particle trajectories, rendering the effective dynamics dissipative. The active (segregating) particles experience modified velocities due to segregation forces. The motion of the active particles is the sum of two velocities $\mathbf{v}^{\text{active}}(\mathbf{x},t) = \mathbf{v}^{\text{flow}}(\mathbf{x},t) + \mathbf{v}^{\text{drift}}(\mathbf{x},t)$, where $\mathbf{v}^{\text{drift}}(\mathbf{x},t)$ represents the segregation-induced velocity difference. Since $\nabla\cdot\mathbf{v}^{\text{flow}} = 0$, the volume contraction associated with segregation depends entirely on $\nabla\cdot\mathbf{v}^{\text{drift}}$. The simplest model for the drift velocities in cases of density-driven segregation is:

- Heavy particles: $\mathbf{v}_{\text{h}}^{\text{drift}} = -v_0(1-c_{\text{h}})\hat{y}$
- Light particles: $\mathbf{v}_{\text{l}}^{\text{drift}} = +v_0 c_{\text{h}}\hat{y}$,

where $c_{\text{h}}(\mathbf{x},t)$ is the local concentration of heavy particles and v_0 is positive. The divergence becomes: $\nabla\cdot\mathbf{v}_{\text{h}}^{\text{drift}} = -v_0\partial c_{\text{h}}/\partial y$ for heavy particles and $\nabla\cdot\mathbf{v}_{\text{l}}^{\text{drift}} = +v_0\partial c_{\text{h}}/\partial y$ for light particles. This is the crucial result: the system is volume contracting for the light particles wherever there are concentration gradients.

The key insight is that segregation breaks the volume-preserving property of the underlying flow, creating attractors precisely in the regular regions where the underlying dynamics is most predictable and stable [59]. As a result, attractors emerge within non-chaotic regions, particles of a given species accumulate preferentially in specific zones (often large or dense particles in low-shear

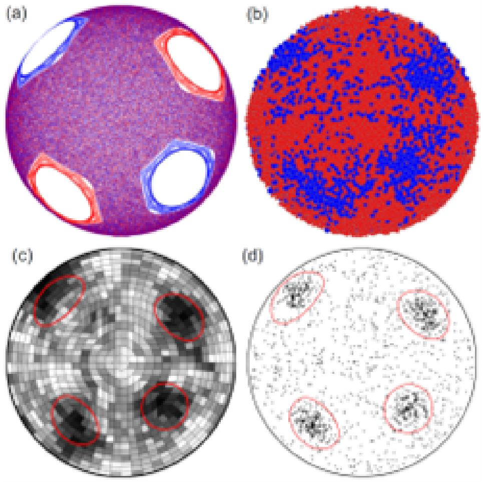

Fig. 12 Segregation in 3D flow in a spherical tumbler showing connections between (a) non-chaotic dynamical islands in an otherwise chaotic flow visualized using a Poincaré map; (b) segregation patterns in a DEM simulation of a mixture of 2 mm (red) and 4 mm (blue) spheres and (c) mean large particle concentration; (d) continuum model incorporating drift. Figure from [59] and used with permission.

regions), and spatial patterns align with dynamical structures such as Kolmogorov–Arnold–Moser (KAM) islands, see Fig. 12. Experiments, continuum descriptions, and dynamical systems analyses show strong agreement: advection by the flow combined with segregation drift leads to predictable, geometry-dependent patterns. This supports a general principle: segregation converts kinematic structures of the flow into material structures of the mixture.

Free-surface shear flows—chutes, heaps, and avalanches—provide a complementary class of systems in which the kinematics are less dominant and constitutive effects play a more central role. The primary outcome is a bulk separation of species driven by kinetic sieving and squeeze expulsion, but the resulting states typically lack sharp spatial boundaries, reflecting a balance between segregation, collisional diffusion, and advection [140], see Fig. 13 for examples of segregation in a bounded heap flow. In steady heap and chute flows, the shear profile localizes segregation near the free surface while diffusion acts throughout the depth, producing smooth, asymmetric concentration profiles [12]. In avalanching and unsteady heap flows, segregation occurs in intermittent surface layers during discrete flow events, leading to incremental reorganization, stratified deposits, and, in some cases, inverse grading [98,141,142].

Importantly, in these free-surface systems segregation can feed back on the flow. Size-dependent concentration variations modify local rheology, giving rise to secondary structures such as particle-size–segregated fronts, levee-channel morphologies, and fingering instabilities in geophysical contexts. These systems are notable for their relative predictability: continuum transport models incorporating segregation fluxes and diffusion capture observed behavior quantitatively, but only when constitutive dependencies on shear rate, pressure, and particle properties are properly accounted for.

Axial segregation in rotating cylinders constitutes a class by itself and represents an intermediate case. Following rapid radial segregation, alternating axial bands of large particle and small-particle enriched regions emerge over much longer timescales, see Fig. 2, followed by slow coarsening through band merging and migration [143]. While the existence of banding is robust, the mechanisms governing its onset were only recently connected to the granular analog of the Rayleigh-Taylor instability [118], and characteristic length and time scales depend sensitively on operating conditions. These systems thus combine robust macroscopic

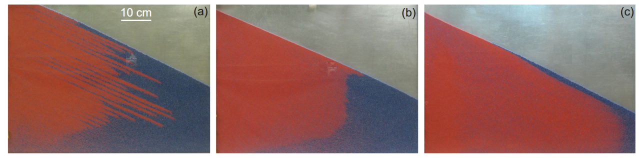

Fig. 13 Final particle configurations of size-bidisperse granular mixtures (blue large and red small particles) in quasi-2D bounded heap flow showing (a) stratification at a low feed rate; (b) streamwise segregation at an intermediate feed rate; and (c) nearly complete mixing at a high feed rate (segregation at the top of heap is due to the transient reduction in flow at the end of the experiment). Figure from [140].

organization with unresolved mechanistic detail.

At the minimal end of complexity, simple shear configurations demonstrate that segregation does not require geometric complexity: even uniform shear between can produce segregated layers [144,145]. However, outcomes in this, the simplest of systems are highly sensitive to particle properties and local conditions, making them useful for probing constitutive laws while also highlighting current limitations in predictive capability.

Across these classes of flows, a unifying perspective emerges. Segregation outcomes reflect an interplay between kinematic structure and constitutive detail. In the baseline case of steady rotation in a circular container, kinematics are simple and outcomes are correspondingly constrained. When chaos is introduced—through geometry or time-dependent forcing—kinematics again dominate, organizing segregation into patterns aligned with dynamical structures. In contrast, in chute and avalanching flows, constitutive dependencies play a central role in determining both steady states and transient evolution. Different systems thus weight kinematics and constitutive physics differently, but it is their interaction that ultimately governs the observable outcomes of segregation.

The examples above suggest a broader lesson: segregation cannot be understood as a mechanism acting in isolation, but as a mechanism filtered through flow kinematics. The same local processes—percolation, kinetic sieving, squeeze expulsion—can produce sharply different outcomes depending on whether the underlying flow is integrable, weakly perturbed, or chaotic. In simple flows, segregation appears as a predictable drift superimposed on regular trajectories; in complex flows, it competes with stretching and folding, leading to patterns that reflect both transport and structure. This interplay highlights a central challenge for modeling: capturing segregation requires not only constitutive descriptions of particle-level mechanisms, but also an understanding of how those mechanisms are organized by the global flow field. In this sense, the outcomes of segregation provide a bridge between local physics and system-scale behavior—precisely the bridge that the different approaches in this review seek to construct.

While recurring structures reveal the presence of underlying mechanisms, they do not by themselves establish how those mechanisms should be represented or combined within a predictive framework. That step requires integration—bringing together conceptual models and computational or continuum descriptions in a way that preserves both interpretability and quantitative accuracy. We turn to one such example next.

3.2 Integrative Synthesis: Convergence Across Representations. The four modeling paradigms for understanding segregation described above each offer unique viewpoints and challenges. In some cases, combining features of one with another has helped to expand the view while reducing the challenges. This overlap is most readily apparent when it comes to developing closure models (constitutive relations) for continuum models of granular segregation, including the Savage and Lun kinetic sieving model combining statistical and continuum mechanical approaches to deter-

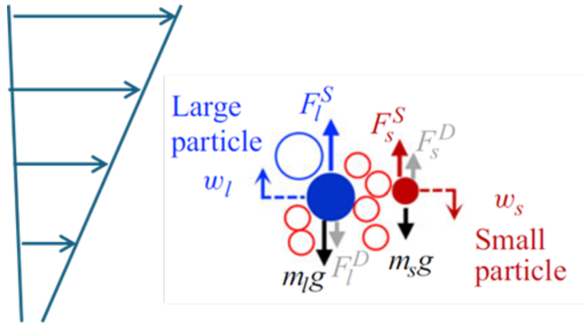

Fig. 14 Free body diagram of gravitational and contact-derived segregation (F_i^S) and drag (F_i^D) forces on a large particle and a small particle in a shear flow. The contact forces result from interactions with the effective medium associated with the flow of the other particles. Figure from [151] and used with permission.

mine the segregation velocity [95]. A recent but novel approach combining insights from DEM simulation, conceptual modeling, and continuum approaches further illustrates this type of synthesis [44,146–151].

Consider the seemingly simple picture in Fig. 14: if the forces acting on an individual large or small particle due to interactions with the surrounding particles are known, the segregation velocity can be predicted. Developing this idea yields a segregation flux, and doing so requires DEM simulation. This is perhaps the clearest example in this review of conceptual insight and computation working not in sequence but in genuine partnership.

A fluid mechanics analogy serves to launch the problem. The terminal velocity of a sphere falling through a viscous liquid follows from a force balance between weight, buoyancy, and Stokes drag. The granular analog, shown in Fig. 14 for a large and a small particle in a bidisperse shear flow, is structurally identical: weight, contact-driven segregation, which is analogous to buoyancy in the fluid problem, and drag forces determine the intruder's terminal, i.e. segregation, velocity. The analogy is illuminating but incomplete, and the incompleteness is instructive.

In a fluid, the drag law and buoyancy are well characterized analytically. In a granular shear flow, an intruder remains locked in a static bed unless energy — here, shear — is continuously supplied. That same shear both liberates the particle and complicates the forces acting on it in ways the conceptual picture alone cannot anticipate. The buoyancy-like segregation force carries both kinematics- and gravity-dependent terms [147], and the granular drag law must be measured rather than derived. DEM simulation is the instrument best suited to that measurement: it can isolate individual force contributions, vary parameters systematically, and supply the constitutive information the continuum model requires.

The result is a representation that neither pure simulation nor pure theory could achieve alone. The conceptual model specifies the right question — what forces act on an intruder, and how do they balance? — and supplies the mathematical structure into which DEM-measured quantities are placed. Simulation fills in what the picture cannot derive. The resulting segregation velocity can then be used in the continuum ADS formalism of Eq. 4. The final outcome is a generalized segregation flux model with genuine predictive reach [150,151] that explains in terms of forces acting on an individual particle the constitutive relations driving quantitative progress in the field over the past two decades [10,11,93].

This example is, in miniature, the argument of the entire review. DEM can tell us what happens; conceptual models tell us why; and the combination, each sharpening the other, is how understanding advances.

The example considered in this section illustrates a broader point: mechanisms are not owned by any single framework. They are expressed differently in particle simulations, continuum equations, statistical formulations, and conceptual models, but they persist across these representations. Understanding arises when these expressions are brought into correspondence—when kinematics is interpreted through structure, and structure is traced back to mechanism.

This perspective suggests a shift in emphasis for the field. The goal is not only to improve predictive accuracy within a given model, but also to identify features that remain consistent as one moves between levels of description. Granular segregation thus becomes more than a collection of phenomena or models; it offers a setting in which such cross-representation invariants can be sought. We take up this challenge in the final section.

3.3 Dimensional Analysis: Reduction as a Mode of Understanding.

This may be an appropriate place to make explicit an idea that has appeared implicitly throughout the preceding sections. Dimensional analysis occupies a distinctive place among the tools available for understanding complex systems. Its power lies not in adding detail but in removing it: by identifying the combinations of variables that govern behavior, it compresses apparent complexity without discarding governing structure. The outcome is a reduced set of dimensionless groups expressing the competition among physical processes. These groups function as organizing coordinates – identifying regimes, exposing dominant balances, and revealing similarities among systems that may differ enormously in scale or material realization.

Classical mechanics is filled with such examples. The Reynolds number expresses the competition between inertial and viscous forces; the Péclet number compares advective and diffusive transport; the Knudsen number marks the transition between continuum and molecular descriptions. Dimensionless groups are compact representations of competing mechanisms.

Granular flows and segregation present several examples. The inertial number I in Sec. 1—comparing the macroscopic deformation timescale to the microscopic inertial timescale—distinguishes quasi-static, dense, and rapid flow regimes represent one such example. Likewise, the experimentally observed scaling of granular diffusivity, $D \sim d^2\dot{\gamma}$, identifies relative motion induced by shear as the dominant transport mechanism in dense granular flows.

Continuum descriptions of segregation often reduce to a competition between segregation fluxes, collisional diffusion, and convective effects yielding reduced parameters that determine whether mixtures sharpen into segregated bands, remain mixed, or approach intermediate steady states. An example of these parameters comes from the ADS equations (Eq. 4) from which two dimensionless parameters naturally arise: a segregation based parameter that represents the ratio of the advection time scale to the segregation time scale and the Péclet number that represents the ratio of the collisional diffusion time scale to the advection time scale [82].

Here we consider only dry dense granular flows, but there exist many parallels between dry granular flows and a wide range of multiphase flows, most notably particulate suspensions. The transition between the literatures on suspensions and granular dynamics ought to be smoother. Many systems classified as "granular" and many classified as "suspensions" differ less by containing fundamentally distinct physics than by differences in emphasis, methodology, and modeling tradition. Dense suspensions, slurries, debris flows, and sedimenting particles all occupy overlapping physical territory. The suspension literature emerged grounded in fluid mechanics, statistical physics, and transport theory, where nondimensionalization, scaling, and asymptotic reasoning play central organizing roles (see, e.g., Guazzelli and Morris [152,153]). Granular physics, especially in its earlier development, evolved more strongly around phenomenology and collective behavior. Yet the distinction between liquid-immersed granular systems [154] and suspensions often reflects differences in dominant interactions and disciplinary perspective more than fundamentally distinct physics. In this sense, the scaling-oriented style characteristic of suspension mechanics may have much to contribute to the continuing development of granular segregation.

Reduced descriptions of this kind do not emerge automatically from simulation output; they require a deliberate act of identification. This points to a complementarity worth making explicit. Computation expands descriptive capacity; dimensional analysis reduces descriptive complexity. One adds detail, the other identifies organization. The two are therefore not in competition but mutually reinforcing. Simulation can reveal which variables matter empirically; dimensional analysis provides the framework for understanding why they matter and how their effects scale.

Increasing computational power does not diminish the importance of reduced descriptions; it increases the need for them. In granular segregation, where the governing particle-scale laws are largely known yet collective behavior remains difficult to explain, the identification of organizing dimensionless structures remains central to principle-based understanding.

Dimensional analysis therefore occupies a distinctive epistemic role: it is neither full theory nor mere approximation, but a method for discovering the reduced structures through which complex systems become intelligible.

Dimensional reasoning operates across all four perspectives considered in this review. At the particle level it identifies the characteristic scales governing interactions. In continuum descriptions it exposes dominant balances among the terms of governing equations. In statistical approaches it clarifies the parameters controlling transitions between collisional and enduring-contact regimes. Conceptually, it functions as a mechanism of abstraction: a way of asking which aspects of a phenomenon are essential and which are incidental to the physics at hand. In this sense, a dimensionless group is not merely a notational convenience – it is a statement about what a system knows about itself.

4 PERSPECTIVE AND OUTLOOK: Toward Principle-Based Understanding

We began this review mentioning limitations and *de novo* contributions, while Secs. 2 and 3 have shown that granular segregation can be described through multiple, formally distinct frameworks that nevertheless encode common underlying mechanisms. The question that follows is whether these recurring structures can be elevated to the status of principles—statements that hold across systems, models, and scales. This shift from correspondence between models to identification of invariants defines the agenda for the field going forward.

Granular segregation in dense flows presents a paradox that has only sharpened with time. At the particle scale, the governing laws are known: interactions are local, forces are well characterized, and Newton's equations are exact. At the macroscopic scale, however, the emergence of robust, system-spanning segregation patterns remains only partially understood. This gap is not due to a lack of data—indeed, modern simulation can generate it in overwhelming abundance—but to a lack of sufficiently incisive description that is only now being developed.

This review has argued that progress in understanding granular segregation has not followed a single trajectory but rather has emerged from the interplay of four complementary approaches: particle-based simulation, continuum modeling, statistical mechanics, and conceptual models. These are not competing paradigms but different resolutions of the same problem. Each captures certain aspects of the phenomenon while obscuring others.

Some frameworks are more generative as metaphors than as formalisms — they illuminate by analogy but resist the literal extension that their elegance seems to invite. The very power of any formalism can become a trap: when a framework has proven itself generative, when it has delivered insight and built a community around it, the natural impulse is to push further. The kinetic theory of granular flow is a case in point — the Boltzmann framework is seductive precisely because it works so beautifully for molecular gases, and its partial successes in the granular setting make retreat difficult to contemplate. The history of Wilson's renormalization group theory [155] offers a wider-angle view of the same phenomenon. After its extraordinary triumphs in quantum field theory and critical phenomena, it was carried — sometimes carefully, sometimes not — into turbulence, economics, and neuroscience. Occasionally the transplant took; more often it produced a literature that resembled rigor without quite achieving it. The framework, originally a tool, had become an aspiration.

A central conclusion is that prediction alone is not understanding. Discrete element methods can now reproduce increasingly complex segregation phenomena with high fidelity. Yet the ability to simulate does not, by itself, reveal the governing structure of the phenomenon. The role of conceptual models—often deliberately simplified, sometimes *ad hoc*—is therefore not diminished by computational advances but amplified. They provide the means to isolate mechanisms, to identify dominant balances, and to construct explanations that generalize beyond specific cases.

From this perspective, segregation outcomes can be understood as reflecting an interplay between kinematic structure and constitutive detail. The flow field—whether steady, time-periodic, or geometrically induced—sets the stage by organizing trajectories, interfaces, and mixing pathways. Material properties (size, density, shape, friction) then determine how particles respond within that structure. Different systems weigh these factors differently. In chute and avalanching flows, for example, segregation is strongly tied to shear-driven mechanisms such as kinetic sieving and squeeze expulsion, operating within a gravity-imposed kinematic framework. In rotating tumblers, by contrast, geometric constraints and flow periodicity can generate complex, even chaotic, advection patterns that strongly influence segregation outcomes.

Several directions emerge as particularly important for future work:

1. Beyond closure: identifying replacements for classical assumptions. Continuum and statistical approaches rely on assumptions—molecular chaos, scale separation, local rheology—that are systematically violated in dense, contact-dominated flows. A central open question is not simply how to extend these frameworks, but what replaces their foundational assumptions. Progress here will likely require hybrid approaches that combine elements of particle-level resolution with reduced-order descriptions.

2. Coupling between flow and composition. Most current models treat the velocity field as given, with segregation evolving within it. Yet increasing evidence shows that concentration fields can, in turn, modify rheology, packing fraction, and stress transmission. Fully coupled descriptions—where flow and segregation co-evolve—remain in their early stages [109,110,115], particularly for mixtures with large size or density contrasts.

3. Role of geometry and time dependence. Segregation is often treated as a local process driven by gradients. However, global features—container geometry, boundary conditions, and time-periodic forcing—can induce complex flow structures, including chaotic advection, that fundamentally alter segregation pathways. Understanding how such kinematic features interact with local segregation mechanisms remains an open and largely underexplored area.

4. Large size ratios and non-monotonic behavior. Recent results point to phenomena that challenge established intuition, including layering where segregation is expected [142,145], segregation velocities non-monotonic with shear rate [120], and strong concentration-dependent changes in packing [117]. Some of these phenomena appear to be associated with which species (large or small) constitutes the continuous or dispersed phase [156], and further suggest that existing models, largely developed for modest size ratios, are unlikely to extrapolate reliably to more extreme regimes.

5. Conceptual synthesis across scales. Perhaps the most important direction is methodological rather than technical: the deliberate integration of conceptual models with computational and continuum approaches. Some of the most successful recent advances—such as force-balance models for intruder segregation—have emerged precisely from this synthesis. Developing such hybrid frameworks in a systematic way remains a key opportunity.

More broadly, granular segregation illustrates a recurring pattern in science. When systems are simple, reductionist descriptions suffice. When they become complex, statistical and continuum approaches provide tractable alternatives. But when these, too, encounter limits—as in segregating dense granular flows—progress often depends on the reintroduction of conceptual, mechanism-driven models that cut across levels of description. In this sense, conceptual models are not a preliminary stage to be discarded, but a persistent and essential component of scientific reasoning to be preserved.

The field is now in a position where all four approaches—particle, continuum, statistical, and conceptual—are mature enough to inform one another. The challenge is not to choose between them, but to understand their relationships and to deploy them in combination. Achieving this synthesis is likely to define the next phase of progress toward a principle-based understanding of granular segregation.

This article focused on a relatively narrow set of questions within the vast landscape of physics and engineering science. Yet it is worth asking whether broader lessons may follow. It is instructive to return to figures invoked at the outset – Maxwell, Poincaré, and Boltzmann. Maxwell advanced kinetic theory at a time when the existence of molecules was not yet firmly established. For much of his career, Poincaré held that the atom was a well-crafted hypothesis rather than a confirmed reality – a stance he qualified, though never fully abandoned, after Perrin's Brownian motion experiments. Boltzmann fought bitterly for the reality of atoms and he did not live to see the argument vindicated: he died in 1906, just as the experimental vindication he needed was arriving. What they could not observe, they had to imagine.

We now inhabit a very different scientific landscape where we can probe matter at extraordinary resolution and compute the dynamics of vast numbers of particles. But this raises a fundamental question: does increased capability diminish the need for imagination? The answer suggested here is no. If anything, it makes it more necessary. The plea of this article is that explanation, true understanding, still requires acts of conceptual invention that computation alone cannot supply.

We close with a remark on the spirit in which this review was written. Its structure and arguments emerged from many conversations—and many disagreements—within our group. We have come to believe that productive intellectual work depends less on consensus than on sustaining the right tensions. The four lenses we have proposed are our current attempt to keep those tensions in view. We would be pleased if readers find in them not conclusions, but provocations—ideas that may extend the field in directions that belong to the field, not to us.

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